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Harmonic analysis of distortion in amplitude modulation systems with asymmetrical sidebands

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RESEARCH DEPARTMENT

HARMONIC ANALYSIS OF DISTORTION IN AMPLITUDE MODULATION SYSTEMS WITH ASYMMETRICAL SIDEBANDS

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(RA-48)

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SUMMARY

The general case of two modulating tones is mainly considered, and the amplitudes of the various frequency components appearing at the output of an ideal envelope detector are evaluated, including cases with unequal upper and lower sidebands. Difficulties associated with overmodulation are discussed, and a general method of finding the magnitude of any distortion product is explained. This method is not affected in a particular case by the modulation depth.

Coefficients associated with the lower-order distortion products have been evaluated (Table 1) for cases in which the sidebands have the ratio 1, ½, 1/3 or 1/9, and for the single-sideband case. For comparison between theoretical and experimental results, a measurable quantity has been calculated; agreement is remarkably good.

1. INTRODUCTION

Ideally, an analysis of a multitone modulating system of n tones would be valuable, but the complexity of such an analysis increases violently with n. The case $n = 1$ is special, and in what follows we have therefore taken n to be 2, and given special attention to cases where there is symmetry, or any other feature which simplifies the analysis.

The main difficulty of the analysis is that the output from an envelope detector is such that only its magnitude, $S(t)$, is determinate; the detector output is essentially positive. It is necessary to take a square root in the process of determining $S(t)$, and the positive square root must be chosen. Now the positive square root of x^2 is not x unless we are certain that x is always positive; in general, the positive square root is $|x|$, and this expression has an awkward singularity when x is capable of changing sign. In the case of $S(t)$, this difficulty can arise whenever there is over-modulation.

In Section 2, $S(t)$ is formulated implicitly for the general case, while in Section 3 certain cases when $S(t)$ is simple and explicit (specified by equation 19 below), and other cases without overmodulation are discussed. It appears to be advantageous in these cases to keep the modulation depth fixed and vary the relative amplitude of the two sidebands. The complications introduced by overmodulation and a way to overcome them are discussed in Section 4.

Only cases in which the upper and lower sidebands are initially in phase with the carrier are considered, since these are the cases of greatest practical importance, and many cases of arbitrary initial sideband phase differences are very similar to related cases without such phase differences. Theoretical results obtained are discussed in Section 5 and summarized in Table 1.

In Section 6 some theoretical results obtained are expressed in terms of a readily measurable quantity and compared with experimental results. Section 7 gives conclusions.

2. THE DERIVATION OF $S(t)$ IN THE GENERAL TWO-TONE CASE

An expression for a carrier-wave modulated with a signal of n tones is

$$E(t) = \left[1 + \sum_{i=1}^{n}\left(a_i \cos \omega_i t + b_i \cos \omega_i t\right)\right] \cos \omega_0 t \tag{1}$$

$$+ \left[\sum_{i=1}^{n}\left(a_i \sin \omega_i t - b_i \sin \omega_i t\right)\right] \sin \omega_0 t \qquad (1)$$

where

ω_0 is the (angular) carrier frequency
ω_i $(i = 1, 2 \ldots n)$ are the (angular) frequencies of the modulating tones,
a_i is the amplitude of the lower sideband component at frequency ω_i
b_i is the amplitude of the upper sideband component at frequency ω_i

and it is assumed that initially both upper and lower sidebands are in phase with the carrier.

The output $S(t)$ to be expected from an ideal envelope detector is given by

$$S(t) = \left(\left\{ 1 + \sum_{i=1}^{n} \left[a_i \cos \omega_1 t + b_i \cos \omega_i t \right] \right\}^2 + \left\{ \sum_{i=1}^{n} \left[a_i \sin \omega_i t - b_i \sin \omega_i t \right] \right\}^2 \right)^{1/2} \qquad (2)$$

As already indicated, we shall be concerned only with the case in which $n = 2$. We shall also devote our main attention to cases in which the conditions (i) ω_1 and ω_2 are large compared to their difference, (ii) $a_1 = a_2$ and (iii) $b_1 = b_2$ are all fulfilled. When $n = 2$, the modulation depth M is given as a percentage by

$$M = 100\,(a_1 + a_2 + b_1 + b_2) \qquad (3)$$

Consider first the case in which $\omega_1 = P\Omega$, $\omega_2 = Q\Omega$, where P and Q are integers without common factor. Then $S(t)$ given by equation (2) is a periodic function of t having period $2\pi/\Omega$, and this is expressible as a Fourier series

$$S(t) = \tfrac{1}{2}\,\alpha_0 + \alpha_1 \cos \Omega t + \alpha_2 \cos 2\Omega t + \ldots \qquad (4)$$

which has only cosine terms. Suppose that $P = 4$ and $Q = 5$ (a particular case which is relevant to the comparison of experimental results with theoretical results summarized in Table 1 (Section 5) and discussed in Section 6). A distortion product will have a frequency of the form $(p\omega_1 + q\omega_2)$, where p and q are integers; for the particular case, the distortion products will therefore have frequencies $s\Omega$ where

$$4p + 5q = s \qquad (5)$$

p and q may be positive or negative subject to the condition $s \geqslant 0$. The important terms of the particular Fourier series for $S(t)$ defined by equations (2) and (4) are in general those for which $|p| + |q|$ is small, rather than those for which s is small. To emphasize this, it is therefore desirable to specify the coefficient of $\cos s\Omega t$ in (4) as $\alpha_{p,q}$ rather than as α_s. The first eleven terms of the Fourier series for $S(t)$ when $P = 4$, $Q = 5$ are thus

$$\begin{aligned} S(t) = {} & \tfrac{1}{2}\,\alpha_{0,0} + \alpha_{-1,1} \cos\Omega t + \alpha_{-2,2}\cos 2\Omega t + \alpha_{2,-1}\cos 3\Omega t \\ & + \alpha_{1,0}\cos 4\Omega t + \alpha_{0,1}\cos 5\Omega t + \alpha_{-1,2}\cos 6\Omega t \\ & + \alpha_{3,-1}\cos 7\Omega t + \alpha_{2,0}\cos 8\Omega t + \alpha_{1,1}\cos 9\Omega t \\ & + \alpha_{0,2}\cos 10\Omega t + \ldots\ldots \end{aligned} \qquad (6)$$

We expect the most important terms here to be the d.c. term and those in $4\Omega t$, $5\Omega t$, Ωt, $8\Omega t$, $9\Omega t$ and $10\Omega t$. Similar considerations apply for any other values of P and Q. A few specific cases investigated numerically indicated that although a small change in the value of ω_2/ω_1 may be associated with a large change in P, Q and s for a given value of p and q, the change in $\alpha_{p,q}$ is small for fixed a_1, a_2, b_1 and b_2. For this reason, the case in which ω_2/ω_1 is an 'incommensurable' quantity like $\sqrt{2}$ which theoretically cannot be expressed as the ratio of two integers has been ignored altogether, and the main interest is in cases where P and Q are small; the simplest case is when $Q - P = 1$ so that Ω equals $\omega_2 - \omega_1$ instead of a fraction thereof. Now equation (6) is tied to the particular values $P = 4$ and $Q = 5$, but in the general case we can still rearrange (6) so that the terms likely to be most important come first, and write

$$\begin{aligned} S(t) = {} & \tfrac{1}{2}\,\alpha_{0,0} + \alpha_{1,0}\cos P\Omega t + \alpha_{0,1}\cos Q\Omega t \\ & + \alpha_{-1,1}\cos(Q-P)\Omega t + \alpha_{2,0}\cos 2P\Omega t + \alpha_{1,1}\cos(P+Q)\Omega t \\ & + \alpha_{0,2}\cos 2Q\Omega t + \ldots \end{aligned} \qquad (7)$$

An obvious way of trying to determine the quantities $\alpha_{p,q}$ appears to be to square $S(t)$ in (7) and compare this with the squared right-hand-side of (2), equating coefficients of $\cos\,(p\omega_1 + q\omega_2)\,t$ for various values of the integers p and q, taken in ascending order of $|p| + |q|$. The terms actually written down in equation (7) are those for which this quantity does not exceed 2. Terms for which $|p| + |q|$ exceeds 2 will of course contribute to the coefficients associated with terms for which $|p| + |q| \leqslant 2$, but in general we expect such contributions to be small, and we start from a case in which they are zero in Section 3.

Since the equations determining the quantities $\alpha_{p,q}$ are obtained by squaring $S(t)$ in equation (7), the equations would equally well be satisfied if every $\alpha_{p,q}$ was replaced by $-\alpha_{p,q}$. Now in equation (2), $S(t)$ is essentially positive, and it is also continuous with respect to t. But if, in equation (7), $\alpha_{0,0}$ is taken as positive, it does not follow that $S(t)$ as given by equation (7) will be positive for all values of t. When the modulation percentage M does not exceed 100, there is no danger that $S(t)$ given by equation (7) can change sign, for in equation (2), $S(t)$ can be regarded as the sum of a fixed vector of magnitude one and four rotating vectors of magnitudes a_1, a_2, b_1 and b_2. If $M < 100$, then not even if all the four vectors are simultaneously in the opposite direction to the fixed vector are they capable of cancelling it completely. When $M \geqslant 100$, such cancellation may or may not occur.

As the comparison of equations (2) and (7) is complicated in the general case, we shall apply our main analysis to the simpler case where

$$a_1 = a_2 = a; \quad b_1 = b_2 = a/k \qquad (8)$$

i.e. the two upper sidebands are equal, and the two lower sidebands are also equal. This case is considered in detail in Section 3 when there is no overmodulation, and in Section 4 when there is overmodulation.

3. THE SPECIAL CASE DEFINED BY EQUATION (8)

If in the special case defined by equation (8) there is no overmodulation, and coefficients $\alpha_{p,q}$ are ignored for $|p| + |q| > 2$, the Fourier series (7) correctly has only cosine terms, and the seven equations giving $\alpha_{0,0}$, $\alpha_{1,0}$, $\alpha_{0,1}$, $\alpha_{2,0}$, $\alpha_{1,1}$, $\alpha_{0,2}$ and $\alpha_{-1,1}$ obtained by comparing the squares of the right hand sides of (2) and (7), are:-

From the d.c. term –

$$1 + 2a^2 \left[1 + (1/k^2)\right] = \tfrac{1}{4}\alpha_{0,0}^2 + \tfrac{1}{2}(\alpha_{1,0}^2 + \alpha_{0,1}^2) + \tfrac{1}{2}(\alpha_{2,0}^2 + \alpha_{1,1}^2 + \alpha_{0,2}^2 + \alpha_{-1,1}^2) \qquad (9)$$

From the term in $\omega_1 t$ –

$$2a \left[1 + (1/k)\right] = \alpha_{0,0}\alpha_{1,0} + \alpha_{1,0}\alpha_{2,0} + \alpha_{0,1}\alpha_{1,1} + \alpha_{0,1}\alpha_{-1,1} \qquad (10)$$

From the term in $\omega_2 t$ –

$$2a \left[1 + (1/k)\right] = \alpha_{0,0}\alpha_{0,1} + \alpha_{1,0}\alpha_{1,1} + \alpha_{1,0}\alpha_{-1,1} + \alpha_{0,1}\alpha_{0,2} \qquad (11)$$

From the term in $2\omega_1 t$ –

$$2a^2/k = \alpha_{0,0}\alpha_{2,0} + \tfrac{1}{2}\alpha_{1,0}^2 + \alpha_{1,1}\alpha_{-1,1} \qquad (12)$$

From the term in $2\omega_2 t$ –

$$2a^2/k = \alpha_{0,0}\alpha_{0,2} + \tfrac{1}{2}\alpha_{0,1}^2 + \alpha_{1,1}\alpha_{-1,1} \qquad (13)$$

From the term in $(\omega_1 + \omega_2)t$ –

$$4a^2/k = \alpha_{0,0}\alpha_{1,1} + \alpha_{1,0}\alpha_{0,1} + \alpha_{2,0}\alpha_{-1,1} + \alpha_{0,2}\alpha_{-1,1} \qquad (14)$$

From the term in $(\omega_2 - \omega_1)t$ –

$$2\alpha^2 \left[1 + (1/k^2)\right] = \alpha_{0,0}\alpha_{-1,1} + \alpha_{1,0}\alpha_{0,1} + \alpha_{2,0}\alpha_{1,1} + \alpha_{1,1}\alpha_{0,2} \qquad (15)$$

Subtracting (13) from (12) –

$$\alpha_{0,0}(\alpha_{2,0} - \alpha_{0,2}) + \tfrac{1}{2}(\alpha_{1,0} + \alpha_{0,1})(\alpha_{1,0} - \alpha_{0,1}) = 0 \qquad (16)$$

Subtracting (11) from (10) –

$$(\alpha_{1,0}\alpha_{2,0} - \alpha_{0,1}\alpha_{0,2}) + (\alpha_{0,0} - \alpha_{-1,1} - \alpha_{1,1})(\alpha_{1,0} - \alpha_{0,1}) = 0 \qquad (17)$$

Equations (16) and (17) are both completely satisfied if both

$$\alpha_{1,0} = \alpha_{0,1} \text{ and } \alpha_{2,0} = \alpha_{0,2} \qquad (18)$$

and we shall assume in what follows that this is the only relevant solution, because of the essential symmetry of equations (8).

If $k = 1$, and $a \leqslant 0{\cdot}25$ to prevent overmodulation, equations (9) to (15) are exactly satisfied by

$$\alpha_{0,0} = 2; \quad \alpha_{1,0} = 2a; \quad \alpha_{2,0} = \alpha_{1,1} = \alpha_{-1,1} = 0 \qquad (19)$$

as expected for the no-distortion condition with symmetrical sidebands. (They are also satisfied by $\alpha_{0,0} = -2$ and $\alpha_{1,0} = -2a$ with other $\alpha_{i,j}$ zero as already mentioned, but this alternative will not be further considered in this Section.) Since $S(t)$ can in general be expected to vary continuously when any of the quantities involved is slightly altered, we can therefore expect $\alpha_{0,0}$ and $\alpha_{1,0}$ to be near these values and $\alpha_{2,0}$, $\alpha_{1,1}$ and $\alpha_{-1,1}$ to be small in neighbouring cases.

If we keep the modulation percentage M constant and vary k when there is no overmodulation, a very useful initial approximation appears to be

$$\alpha_{0,0} \approx 2; \quad \alpha_{1,0} \approx a\left(1 + \frac{1}{k}\right) = M/200 \qquad (20)$$

Substituting this value of $\alpha_{1,0}$ (and $\alpha_{0,1}$) into (9) and neglecting $\alpha_{2,0}^2$, $\alpha_{1,1}^2$, $\alpha_{0,2}^2$ and $\alpha_{-1,1}^2$ gives an improved value of $\alpha_{0,0}$; substituting this into (12) and (14) and (15) gives the approximate values of $\alpha_{2,0}$, $\alpha_{1,1}$ and $\alpha_{-1,1}$. Finally substitution into (10) gives an improved value of $\alpha_{1,0}$ with which the whole process can be repeated. We have to remember, particularly at modulation percentages near 100, that appreciable contributions may be coming in from higher-order terms of frequency $p\omega_1$ and $q\omega_2$ where $|p| + |q| > 2$ and that we therefore cannot expect to achieve high accuracy in all cases in this way, but the exact relation (19) indicates that there are many cases in which high accuracy will be achieved.

4. THE EFFECT OF OVERMODULATION IN SYMMETRICAL CASES

We have already noted that in the case when k was 1 and $a \leqslant 0{\cdot}25$ (so that there is no overmodulation) the exact solution (19) was associated with an ambiguity in sign which was fortunately then irrelevant. Physically, this kind of ambiguity in sign is associated with the possibility of overmodulation, which we must now consider.

If $k = 1$ in equation (8), and $a \leqslant 0{\cdot}25$, then the coefficients of $S(t)$ are given by equations (19), so that

$$S(t) = 1 + 2a \cos \omega_1 t + 2a \cos \omega_2 t \qquad (21)$$

and this equation is true whatever the ratio ω_1/ω_2 may be.

If $k = 1$ and $a > 0{\cdot}25$, however, equation (21) has to be replaced by

$$S(t) = |1 + 2a \cos \omega_1 t + 2a \cos \omega_2 t| \qquad (22)$$

and equation (19) is no longer valid, because there are some ranges of values of t for which

$$(1 + 2a \cos \omega_1 t + 2a \cos \omega_2 t)$$

is positive and some for which this expression is negative. These ranges are separated by values of t for which $S(t)$ is zero, and zeros of $S(t)$ will depend upon the value of the ratio ω_2/ω_1 as well as upon the value of a.

More generally, when there is overmodulation, there is also the possibility that $S(t)$ given by (2) can have zeros which are singularities. This means that equations analogous to the system (9) – (15) are not valid, and that any particular coefficient $\alpha_{p,q}$ will depend upon the ratio ω_2/ω_1 as well as upon a_1, a_2, b_1 and b_2. There is, however, a well known 'summation technique'* for obtaining any required Fourier coefficient in such cases, when the ratio ω_1/ω_2 is the ratio of two integers P/Q which have no common factor, so that $S(t)$ is periodic with period T, say. Suppose that

$$\left.\begin{aligned} S(t) &= U_0 \text{ when } t = 0 \text{ or } t = T \\ S(t) &= U_r \text{ when } t = rT/N \ (r = 1, 2 \ldots N{-}1) \end{aligned}\right\} \qquad (23)$$

and that we seek an approximation of the form

$$S(t) \approx \tfrac{1}{2}\alpha_0 + \sum_{s=1}^{R} \left[\alpha_s \cos \frac{2\pi st}{T} + \beta_s \sin \frac{2\pi st}{T} \right] \qquad (24)$$

where $R < \tfrac{1}{2}(N-1)$. We shall assume that N is sufficiently large which means that α_s and β_s are negligible for $s \geqslant N-R$. Then we can take

$$\left.\begin{aligned} \alpha_s &= \frac{2}{N} \sum_{r=0}^{N-1} U_r \cos \frac{2\pi rs}{N} \\ \beta_s &= \frac{2}{N} \sum_{r=1}^{N-1} U_r \sin \frac{2\pi rs}{N} \end{aligned}\right\} \qquad (25)$$

An Elliott autocode programme has been formulated for the evaluation of equations (25) in specific cases. This programme has been used to determine the coefficients given in Table 1 (Section 5) for cases with and without overmodulation. Most of these calculations were made for $\omega_1/\omega_2 = 4/5$ so that conditions (5) applied for determining the important terms for which $|p| + |q|$ is small. Although the calculation has to be done in terms of the value of s, the importance of the coefficient calculated is primarily controlled by the values of p and q. The coefficients given in Table 1 (Section 5) embodying results actually calculated are therefore specified in terms of p and q by analogy with equation (7), and not in terms of s.

5. THEORETICAL RESULTS

Numerical values of the principal coefficients in the harmonic analysis of $S(t)$ given by equation (2) are tabulated in Table 1 below. It is assumed that $\omega_2/\omega_1 = 5/4$ throughout; only cosine terms are present. k has the same meaning as in equation (8), and has one of the values 1, 2, 3, 9 or ∞ (the single sideband case). $k = 1$ is the equal sideband case, for which (21) is an exact solution when there is no overmodulation. The principal coefficients are deemed to be those listed in equation (7), where for these particular calculations Ω means $(\omega_2 - \omega_1)$.

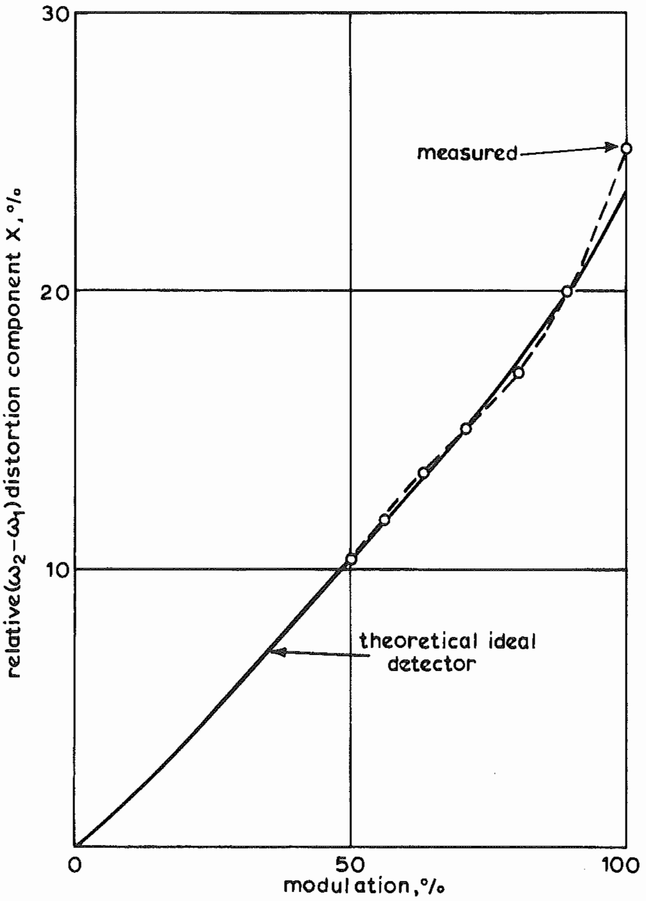

Fig. 1 - Comparison of theoretical and experimental values of $(\omega_2-\omega_1)$ distortion component

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* This is discussed by Clenshaw[1] in terms of Chebyshev polynomials.

TABLE 1

Intermodulation Terms for Two Tones

$a_1 = a_2 = a$; $b_1 = b_2 = a/k$; $M = 200a(k+1)/k$ (see equations (2) and (3)). $\frac{1}{2}\alpha_{0,0}$ is the d.c. term and $\alpha_{p,q}$ is the coefficient of cos $(p\omega_1 + q\omega_2)t$ with $\omega_1 = 4\Omega$ and $\omega_2 = 5\Omega$ (see equation (7))

Modulation depth (M%)	k	a	$\alpha_{0,0}$	$\alpha_{1,0}$	$\alpha_{0,1}$	$\alpha_{2,0}$	$\alpha_{1,1}$	$\alpha_{0,2}$	$\alpha_{-1,1}$
50	∞	0·25	2·0633	0·2440	0·2440	−0·0146	−0·0297	−0·0146	0·0318
	9	0·225	2·0410	0·2460	0·2460	−0·0096	−0·0193	−0·0096	0·0207
	3	0·1875	2·0163	0·2484	0·2484	−0·0038	−0·0077	−0·0038	0·0083
	2	0·1667	2·0073	0·2493	0·2493	−0·0017	−0·0034	−0·0017	0·0037
	1	0·125	2	0·25	0·25	0	0	0	0
70	∞	0·35	2·1257	0·3330	0·3330	−0·0265	−0·0552	−0·0266	0·0634
	9	0·315	2·0827	0·3384	0·3384	−0·0177	−0·0364	−0·0176	0·0421
	3	0·2625	2·0334	0·3451	0·3451	−0·0072	−0·0147	−0·0072	0·0172
	2	0·2333	2·0150	0·3478	0·3478	−0·0033	−0·0066	−0·0033	0·0078
	1	0·175	2	0·35	0·35	0	0	0	0
90	∞	0·45	2·2121	0·4120	0·4120	−0·0387	−0·0836	−0·0387	0·1077
	9	0·405	2·1427	0·4228	0·4228	−0·0260	−0·0556	−0·0260	0·0737
	3	0·3375	2·0597	0·4376	0·4376	−0·0108	−0·0227	−0·0108	0·0316
	2	0·3000	2·0274	0·4441	0·4441	−0·0049	−0·0103	−0·0049	0·0146
	1	0·225	2	0·45	0·45	0	0	0	0
100	∞	0·5	2·2659	0·4457	0·4457	−0·0431	−0·0965	−0·0432	0·1360
	9	0·45	2·1818	0·4599	0·4599	−0·0287	−0·0639	−0·0288	0·0950
	3	0·375	2·0785	0·4804	0·4804	−0·0116	−0·0257	−0·0115	0·0425
	2	0·3333	2·0368	0·4902	0·4902	−0·0052	−0·0114	−0·0050	0·0203
	1	0·25	2	0·5	0·5	0	0	0	0
110	∞	0·55	2·3295	0·4729	0·4729	−0·0443	−0·1057	−0·0443	0·1701
	9	0·495	2·2306	0·4899	0·4899	−0·0278	−0·0678	−0·0279	0·1227
	3	0·4125	2·1065	0·5159	0·5159	−0·0083	−0·0235	−0·0081	0·0600
	2	0·3667	2·0543	0·5297	0·5296	−0·0014	−0·0072	−0·0009	0·0319
	1	0·275	2·0033	0·5470	0·5469	0·0024	0·0026	0·0026	0·0025
130	∞	0·65	2·4820	0·5103	0·5104	−0·0377	−0·1135	−0·0380	0·2531
	9	0·585	2·3554	0·5305	0·5307	−0·0152	−0·0629	−0·0160	0·1952
	3	0·4875	2·1935	0·5623	0·5626	0·0132	−0·0006	0·0126	0·1169
	2	0·4333	2·1233	0·5798	0·5799	0·0245	0·0241	0·0246	0·0804
	1	0·325	2·0513	0·6038	0·6034	0·0328	0·0438	0·0345	0·0395
150	∞	0·75	2·6538	0·5377	0·5376	−0·0288	−0·1184	−0·0290	0·3443
	9	0·675	2·4993	0·5603	0·5605	0·0003	−0·0547	−0·0013	0·2772
	3	0·5625	2·2996	0·5960	0·5970	0·0390	0·0266	0·0361	0·1856
	2	0·5000	2·2117	0·6160	0·6170	0·0558	0·0610	0·0533	0·1423
	1	0·375	2·1198	0·6441	0·6442	0·0709	0·0927	0·0709	0·0922

The coefficients were obtained by means of the computer programme (based on the first of equations (25)) already mentioned. The particular ratio $\omega_1/\omega_2 = 4/5$ was chosen, because it was known that experimental results were available for the case $\omega_1/2\pi = 500$ Hz and $\omega_2/2\pi = 621$ Hz. The good agreement shown in Fig. 1 between experimental and theoretical results (and discussed in Section 6) is a useful and encouraging confirmation of the idea that a particular coefficient $\alpha_{p,q}$ is not very sensitive to small changes in the ratio ω_1/ω_2 when p, q, a_1, a_2, b_1 and b_2 remain constant, when there is no overmodulation. Equations (25) permit the calculation of any desired coefficient by means of the computer programme, whether there is overmodulation or not. But the machine time required for such a calculation rises sharply with N, the number of values of t for which $S(t)$ must actually be calculated. If α_{pq} is not very sensitive to changes in the ratio

ω_2/ω_1, it is clearly advantageous to take this ratio as $(\lambda+1)/\lambda$ where λ is a small integer, if this is at all possible, because N can then be relatively small.

In Table 1 we have confined our attention to cases satisfying equation (8), so that there appears to be a significant loss of generality. The cases tabulated, however, appear to be the most important ones in practice, and many of the cases which do not satisfy equation (8) can be satisfactorily related to cases which do satisfy this equation. There is no restriction on the validity of equations (25) when $S(t)$ is of the form (2), so that any required case can be calculated. The above considerations merely indicate that there is no point in actually carrying out such calculations in the absence of a specific practical need for them.

6. EXPRESSING RESULTS OBTAINED BY CALCULATION IN TERMS OF MEASURABLE QUANTITIES

A quantity which can be readily measured is

$$X = 100\,\frac{\text{average modulus of } (\omega_2-\omega_1) \text{ distortion component}}{\text{average modulus of total output signal}} \qquad (26)$$

We require a theoretically determined quantity which is comparable with X for cases in which there is no overmodulation.

In all the cases of interest, the coefficients $\alpha_{1,0}$ and $\alpha_{0,1}$ (in the notation of Table 1) are equal, and they are much greater than any other coefficient. Also, $(\omega_2-\omega_1)$ is a much lower (angular) frequency than any other mentioned in connection with Table 1, so we have regarded the average modulus of the distortion component as approximately equal to the average value of $|\alpha_{-1,1}\sin(\omega_2-\omega_1)t|$ over a period $2\pi/(\omega_2-\omega_1)$, namely $(2/\pi)|\alpha_{-1,1}|$. The average modulus of total output signal is correspondingly taken as average value of

$$|\alpha_{1,0}\cos\omega_1 t + \alpha_{0,1}\cos\omega_2 t + \alpha_{-1,1}\cos(\omega_2-\omega_1)t| \qquad (27)$$

over the same period. This average modulus of total output signal was calculated by means of a computer programme when $\omega_1 = 0{\cdot}8\,\omega_2$, so that the condition for the total output signal to be zero reduced to an equation of degree 5 in $\cos(\omega_2-\omega_1)t$, which could be solved by means of a standard library subroutine.

Fig. 1 shows experimental values of X for a particular receiver (measured at the output of the envelope detector) under single sideband conditions, for modulation depths between 60% and 100% with $\omega_1/2\pi = 500\,\text{Hz}$ and $\omega_2/2\pi = 621\,\text{Hz}$ (broken curve); the corresponding calculated values are plotted as a continuous curve. The agreement is remarkably good. Some further comparisons between theory and experiment are given in a separate report.[2]

7. CONCLUSIONS

A complete survey of the general two-tone case is perfectly feasible. A computer programme is available for analysing any particular numerical case, with or without overmodulation. The most important cases from the practical point of view — which alone are considered in numerical detail here — are those in which the two upper-sideband amplitudes are equal, the two lower-sideband amplitudes are equal, there are no initial differences of phase between corresponding upper and lower sidebands and the ratio $\omega_2/(\omega_2-\omega_1)$ is a reasonably small integer. Other cases can usually be related closely to the above-mentioned special but important cases. The principal numerical results obtained are given in Table 1.

A close agreement between theoretical and measured results has been obtained for the quantity X specified in equation (26), as indicated in Fig. 1.

8. REFERENCES

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